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Second-order finite difference approximation for inverse determination of thermal conductivity

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Abstract—A second-order finite difference procedure is presented for the inverse determination of the thermal conductivity in a one-dimensional heat conduction domain. In this approach, the thermal conductivity of the material is reconstructed by using the available temperature data at discrete grid points. The accuracy of the computational algorithm is investigated. To confirm the validity of the numerical procedure, various comparative examples are presented. The close agreement between the current results and the exact solutions confirms that the proposed finite difference procedure is an effective technique for the inverse determination of thermal conductivity. The method is applicable for linear and nonlinear spatially dependent, as well as temperature dependent, thermal conductivities. In addition, a key feature of the present technique is that *a priori* knowledge of the functional form for the thermal conductivity is not required. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Precise knowledge of the thermophysical properties for composite materials is essential in many thermal management system analyses. An increasing effort in the past decade has been devoted to expanding our knowledge of material properties due to advanced technological developments in material sciences [1–5]. Specifically, an accurate measurement of thermal conductivity is imperative to achieve an optimal thermal control system.

Inverse determination of the thermal conductivity from measured temperature profiles has been the topic of research by many investigators [6–10]. Most of these studies assumed that the thermal conductivity is only a function of the spatial coordinate [11–13]. However, thermal conductivities are temperature dependent quantities in most practical engineering applications [14, 15]. The contribution of the present work is to present a second-order accurate finite difference procedure for the determination of the thermal conductivity, which can be either a constant or spatially- or temperature-varying quantity. A rigorous analysis is presented to establish the uniqueness conditions for the numerical computational approach.

Recently, Lam and Yeung [16] employed a first-order numerical method to determine the thermal conductivity in a one-dimensional heat conduction domain. This paper extends their earlier results, examining the feasibility of using a higher order finite difference technique to determine the thermal conductivity in a heat conduction domain. The thermal conductivity is assumed to be either a function of the spatial coordinate or temperature. In the past, most studies employed an optimization technique to obtain a least squares approximation of the conductivity function. In this study, a second-order finite difference procedure (a direct approach compared to the least squares technique) is used to discretize the heat conduction equation. This converts the governing partial differential equation to a system of linear equations [17, 18]. As a result, the conductivity function can be obtained by solving the system of linear equations. The advantage of this approach is that no prior information is required on the functional form of the thermal conductivity.

Several heat conduction problems have been tested with the procedure for spatial- and temperature-dependent thermal conductivities. The parameter estimation problem can be either linear or nonlinear. The estimated thermal conductivity is verified by comparing with the exact function to confirm the validity of the method. Furthermore, the order of accuracy of the numerical procedure is discussed.

2. ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

In order to illustrate the basic concepts associated with the proposed finite difference procedure for the inverse determination of the thermal conductivity for a heat-conduction system, it is interesting to study a one-dimensional ($0 \leqslant x \leqslant 1$), time-dependent non-homogeneous problem with heat generation. Figure 1 depicts the one-dimensional region R under consideration. The temperature distribution of the med-

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NOMENCLATURE

C	arbitrary constant	T	temperature [C]
f_m	initial temperature function	x	spatial coordinate [m].
g	heat generation [W m^{-3}]		
k	thermal conductivity [W m^{-1} C^{-1}]	Subscript	
q	heat flux [W m^{-2}]	i	time index
t	time [s]	j	space index.

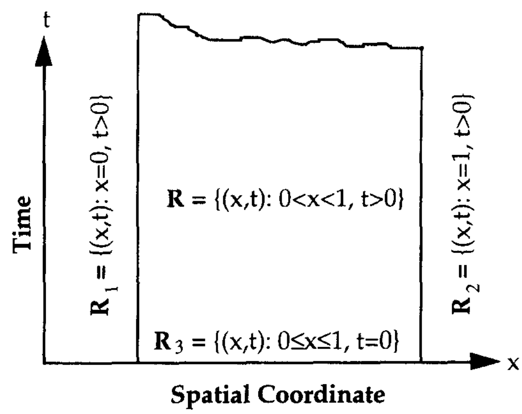

Fig. 1. Region of one-dimensional heat conduction.

ium is initially prescribed over R_3. For times $t > 0$, the boundaries at $x = 0$ and $x = 1$ are subjected to a set of boundary conditions over R_1 and R_2 of the region R, where $R = \{(x, t): 0 < x < 1, t > 0\}$. Everything outside of the region is assumed to be at zero temperature. In addition, the product of the material density and heat capacity is considered as unity. The general one-dimensional heat-conduction equation can be stated as

$$\frac{\partial T(x,t)}{\partial t} - \frac{\partial}{\partial x}\left[k(x,t)\frac{\partial T(x,t)}{\partial x}\right] = g(x,t)$$

$$\text{in } 0 < x < 1, t > 0, \quad (1)$$

where the initial condition is

$$T(x,0) = f_m(x) \quad \text{for } 0 \leqslant x \leqslant 1. \quad (2)$$

For the above second-order partial differential equation, two boundary conditions are required. These boundary conditions can be categorized as follows:

(1) boundary condition of the first kind in which the temperature is prescribed along the boundary surface, such as

$$T(x,t) \text{ is given at } x = 0 \quad \text{or} \quad x = 1 \text{ for } t > 0; \quad (3a)$$

(2) boundary condition of the second kind in which the heat flux is applied at the boundary surface, such as

$$\frac{\partial T(x,t)}{\partial x} \text{ is given at } x = 0 \quad \text{or} \quad x = 1 \text{ for } t > 0; \quad (3b)$$

(3) boundary condition of the third kind in which heat dissipation by convection from a surface to a surrounding environment at zero temperature is specified, such as

$$k(x,t)\frac{\partial T(x,t)}{\partial x} + T(x,t)$$

$$\text{is given at } x = 0 \quad \text{or} \quad x = 1 \text{ for } t > 0. \quad (3c)$$

The main purpose of this study is to determine the conductivity, $k(x, t)$, at any point within the domain $R = \{(x, t): 0 < x < 1, t > 0\}$ with the assumption that the temperature, $T(x, t)$, is known at discrete grid points.

3. INVERSE DETERMINATION OF THERMAL CONDUCTIVITY

In this section, a brief discussion of the requirements which lead to a unique solution of the inverse heat conduction problem is first addressed. Then, a finite difference procedure for the calculation of the thermal conductivity function given the time-dependent temperature distribution, $T(x, t)$, at discrete grid points will be presented.

Necessary condition for the uniqueness of thermal conductivity

Throughout this section, the temperature, $T(x, t)$, is assumed to be known over the entire domain. As a result, derivatives of the temperature (e.g. $\partial T/\partial x$, $\partial^2 T/\partial x^2$ and $\partial T/\partial t$) can also be calculated based on the available temperature profile. The general one-dimensional heat conduction equation (1) can be rewritten as

$$\frac{\partial}{\partial x}\left[k(x,t)\frac{\partial T(x,t)}{\partial x}\right] = \frac{\partial T(x,t)}{\partial t} - g(x,t). \quad (4)$$

Consider the above partial differential equation at $t = \hat{t}$, this nonhomogeneous ordinary differential equation takes the form

$$\frac{\partial}{\partial x}\left[k(x,\hat{t})\frac{\partial T(x,\hat{t})}{\partial x}\right]=\frac{\partial T(x,\hat{t})}{\partial t}-g(x,\hat{t}). \quad (5)$$

The general solution of the nonhomogeneous ordinary differential equation is the sum of the particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation. Hence, let us focus on the homogeneous case,

$$\frac{\partial}{\partial x}\left[k(x,\hat{t})\frac{\partial T(x,\hat{t})}{\partial x}\right]=0. \quad (6)$$

This implies,

$$k(x,\hat{t})\frac{\partial T(x,\hat{t})}{\partial x}=C, \quad (7)$$

where C is an arbitrary constant. If the term $\partial T(x,t)/\partial x$ is nonzero over the entire interval [0, 1], one can rewrite the above equality (7) as,

$$k(x,\hat{t})=\frac{C}{\frac{\partial T(x,\hat{t})}{\partial x}}. \quad (8)$$

Since C is an arbitrary constant, this implies that there are infinitely many solutions for the homogeneous equation (6). Moreover, the implication is that the nonhomogeneous ordinary differential equation (5) also has infinitely many solutions.

Based on the above discussion, a necessary condition for having a unique solution $k(x,t)$ of the ordinary differential equation (5) can be stated as follows. There exists x_0 in the interval [0, 1], such that

$$\frac{\partial T(x_0,\hat{t})}{\partial x}=0. \quad (9)$$

Recall that this discussion is based on the assumption that only the temperature profile is known over the entire domain R. However, this constraint can be relaxed if, for instance, both temperature and surface heat flux at either boundary are known. In such a case, the constant C in equation (8) can be replaced by the surface heat flux. Equation (9) can then be eliminated as the necessary condition for a unique solution. Therefore, three cases will be discussed in detail below for the inverse determination of the thermal conductivity; the first based on temperature data alone, and the second and third on a combination of temperature and surface heat flux at either boundary.

Mathematical formulation

In this section, a numerical procedure based on discrete temperature is presented for the inverse determination of the thermal conductivity function. First, we discretize the entire domain $\{(x,t): x\in[0,1]$ and $t\in[0,\infty)\}$ with mesh width Δx in the spatial direction and Δt in the time direction, grid points $x_j=j\cdot\Delta x$ (where $j=0,1,\ldots,N$ and $N\cdot\Delta x=1$) and $t_i=i\cdot\Delta t$ ($i=0,1,2,\ldots$). The present procedure will assume that the temperature, $T(x,t)$, is known at the grid points, (x_j,t_i).

Now, we will demonstrate the discretization of the governing equation (1). Suppose (x_j,t_i) is an interior point (i.e. $0<j<N$ and $i>0$) and the governing equation is fixed at this particular point; we then have

$$\frac{\partial T^i}{\partial t_j}-\frac{\partial}{\partial x}\left[k\frac{\partial T}{\partial x}\right]_j^i=g_j^i. \quad (10)$$

By applying forward differencing to the time-derivative and expanding the space-derivative, equation (10) can be rewritten as

$$\frac{T_j^{i+1}-T_j^i}{\Delta t}-\frac{1}{4\Delta x^2}[k_{j+1}^i\cdot(T_{j+1}^i-T_{j-1}^i)+4k_j^i\cdot(T_{j+1}^i-2T_j^i+T_{j-1}^i)+k_{j-1}^i\cdot(T_{j-1}^i-T_{j+1}^i)]=g_j^i. \quad (11)$$

Now, the thermal conductivity can be written in terms of the temperature and internal heat generation in the following form:

$$k_{j-1}^i\cdot(T_{j-1}^i-T_{j+1}^i)+4k_j^i\cdot(T_{j+1}^i-2T_j^i+T_{j-1}^i)+k_{j+1}^i\cdot(T_{j+1}^i-T_{j-1}^i)=4\Delta x^2\cdot\left[\frac{T_j^{i+1}-T_j^i}{\Delta t}-g_j^i\right], \quad (12)$$

with the assumption that (x_j,t_i) is an interior point (i.e. $0<j<N$ and $i>0$).

Next, we will discuss the discretization of the governing equation at the boundary. For simplicity, only the boundary R_1 will be considered. Readers can easily extend the discretization to the other boundary (R_2). Suppose (x_j,t_i) is on R_1 (i.e. $j=0$, $i>0$). Then we have the following:

$$\left(\frac{\partial T}{\partial t}\right)_0^i\cong\frac{T_0^{i+1}-T_0^i}{\Delta t}, \quad (13)$$

$$\frac{\partial}{\partial x}\left[k\cdot\frac{\partial T}{\partial x}\right]_0^i\cong\frac{1}{\Delta x^2}[k_1^i\cdot(T_1^i-T_0^i)+k_0^i\cdot(T_2^i-3T_1^i+2T_0^i)]. \quad (14)$$

By substituting equations (13) and (14) into equation (10), the finite difference equation at the boundary surface R_1 takes the form

$$\frac{T_0^{i+1}-T_0^i}{\Delta t}-\frac{1}{\Delta x^2}[k_1^i\cdot(T_1^i-T_0^i)+k_0^i\cdot(T_2^i-3T_1^i+2T_0^i)]=g_0^i, \quad (15)$$

which can be rearranged as

$$k_0^i\cdot(T_2^i-3T_1^i+2T_0^i)+k_1^i\cdot(T_1^i-T_0^i)=\Delta x^2\cdot\left[\frac{T_0^{i+1}-T_0^i}{\Delta t}-g_0^i\right]. \quad (16)$$

Similarly, for the finite difference equation at the boundary surface R_2, one can easily obtain the following equation:

$$k_{N-1}^i \cdot (T_{N-1}^i - T_N^i) + k_N^i \cdot (T_{N-2}^i - 3T_{N-1}^i + 2T_N^i) = \Delta x^2 \cdot \left[\frac{T_N^{i+1} - T_N^i}{\Delta t} - g_N^i\right]. \quad (17)$$

Computational algorithm

The numerical procedure for the inverse determination of the thermal conductivity is summarized below. Suppose we are interested in solving the inverse heat conduction problem at $t = \bar{t}$ by assuming a temperature, $T(x, t)$, which is known only at the grid points. By using equations (12), (16) and (17), one can create the following system of linear equations:

$$\mathbf{Ax} = \mathbf{b}, \quad (18)$$

where $\mathbf{A}$ is an $(N+1) \times (N+1)$ matrix and $\mathbf{x}$ and $\mathbf{b}$ are $(N+1)$ vectors. For simplicity, we assume that the subscript of $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ range from 0 to N as follows:

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & & & & \\ a_{1,0} & a_{1,1} & a_{1,2} & & & \\ & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \\ & & & a_{N-1,N-2} & a_{N-1,N-1} & a_{N-1,N} \\ & & & & a_{N,N-1} & a_{N,N} \end{bmatrix}.$$

$$\mathbf{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_0 \\ \vdots \\ b_N \end{pmatrix}.$$

(1) To set up $\mathbf{A}$ and $\mathbf{b}$. Case (i), temperature data is available—suppose there exists an integer, M, where $0 < M < N$, such that

$$T(x_{M+1}, \bar{t}) - T(x_{M-1}, \bar{t}) = 0 \quad (19)$$

$$T(x_{M+1}, \bar{t}) - 2T(x_M, \bar{t}) + T(x_{M-1}, \bar{t}) \neq 0. \quad (20)$$

The above requirements are equivalent to the necessary condition given by equation (9), so that $dT/dx = 0$ at grid point x_M. For further information see Ref. [16].

For $j = 0$,

$$a_{0,0} = T(x_2, \bar{t}) - 3T(x_1, \bar{t}) + 2T(x_0, \bar{t}), \quad (21)$$

$$a_{0,1} = T(x_1, \bar{t}) - T(x_0, \bar{t}), \quad (22)$$

$$b_0 = \Delta x^2 \left[\frac{T(x_0, \bar{t}+\Delta t) - T(x_0, \bar{t})}{\Delta t} - g(x_0, \bar{t})\right]. \quad (23)$$

For $j = 1, \ldots, N-1$,

$$a_{j,j-1} = T(x_{j-1}, \bar{t}) - T(x_{j+1}, \bar{t}), \quad (24)$$

$$a_{j,j} = 4[T(x_{j-1}, \bar{t}) - 2T(x_j, \bar{t}) + T(x_{j+1}, \bar{t})], \quad (25)$$

$$a_{j,j+1} = T(x_{j+1}, \bar{t}) - T(x_{j-1}, \bar{t}), \quad (26)$$

$$b_j = 4\Delta x^2 \left[\frac{T(x_j, \bar{t}+\Delta t) - T(x_j, \bar{t})}{\Delta t} - g(x_j, \bar{t})\right]. \quad (27)$$

For $j = N$,

$$a_{N,N-1} = T(x_{N-1}, \bar{t}) - T(x_N, \bar{t}), \quad (28)$$

$$a_{N,N} = T(x_{N-2}, \bar{t}) - 3T(x_{N-1}, \bar{t}) + 2T(x_N, \bar{t}), \quad (29)$$

$$b_N = \Delta x^2 \left[\frac{T(x_N, \bar{t}+\Delta t) - T(x_N, \bar{t})}{\Delta t} - g(x_N, \bar{t})\right]. \quad (30)$$

Case (ii), temperature data and heat flux at $x = 0$ are available—in addition to the temperature data, the heat flux, q, is known at $(x_0, \bar{t})$.

For $j = 0$,

$$a_{0,0} = 1, \quad (31)$$

$$a_{0,1} = 0, \quad (32)$$

$$b_0 = \frac{q}{\dfrac{T(x_1, \bar{t}) - T(x_0, \bar{t})}{\Delta x}}. \quad (33)$$

For $j = 1, \ldots, N-1$,

$$a_{j,j-1} = T(x_{j-1}, \bar{t}) - T(x_{j+1}, \bar{t}), \quad (34)$$

$$a_{j,j} = 4[T(x_{j-1}, \bar{t}) - 2T(x_j, \bar{t}) + T(x_{j+1}, \bar{t})], \quad (35)$$

$$a_{j,j+1} = T(x_{j+1}, \bar{t}) - T(x_{j-1}, \bar{t}), \quad (36)$$

$$b_j = 4\Delta x^2 \left[\frac{T(x_j, \bar{t}+\Delta t) - T(x_j, \bar{t})}{\Delta t} - g(x_j, \bar{t})\right]. \quad (37)$$

For $j = N$,

$$a_{N,N-1} = T(x_{N-1}, \bar{t}) - T(x_N, \bar{t}), \quad (38)$$

$$a_{N,N} = T(x_{N-2}, \bar{t}) - 3T(x_{N-1}, \bar{t}) + 2T(x_N, \bar{t}), \quad (39)$$

$$b_N = \Delta x^2 \left[\frac{T(x_N, \bar{t}+\Delta t) - T(x_N, \bar{t})}{\Delta t} - g(x_N, \bar{t})\right]. \quad (40)$$

Case (iii), temperature data and heat flux at $x = 1$ are available—temperature data and surface heat flux, $q(x_N, \bar{t})$, are known.

For $j = 0$,

$$a_{0,0} = T(x_2, \bar{t}) - 3T(x_1, \bar{t}) + 2T(x_0, \bar{t}), \quad (41)$$

$$a_{0,1} = T(x_1, \bar{t}) - T(x_0, \bar{t}), \quad (42)$$

$$b_0 = \Delta x^2 \left[\frac{T(x_0, \bar{t}+\Delta t) - T(x_0, \bar{t})}{\Delta t} - g(x_0, \bar{t}) \right], \quad (43)$$

For $j = 1, \ldots, N-1$,

$$a_{j,j-1} = T(x_{j-1}, \bar{t}) - T(x_{j+1}, \bar{t}), \quad (44)$$

$$a_{j,j} = 4[T(x_{j-1}, \bar{t}) - 2T(x_j, \bar{t}) + T(x_{j+1}, \bar{t})], \quad (45)$$

$$a_{j,j+1} = T(x_{j+1}, \bar{t}) - T(x_{j-1}, \bar{t}), \quad (46)$$

$$b_j = 4\Delta x^2 \left[\frac{T(x_j, \bar{t}+\Delta t) - T(x_j, \bar{t})}{\Delta t} - g(x_j, \bar{t}) \right]. \quad (47)$$

For $j = N$,

$$a_{N,N-1} = 0, \quad (48)$$

$$a_{N,N} = 1, \quad (49)$$

$$b_N = \frac{q}{\dfrac{T(x_N, \bar{t}) - T(x_{N-1}, \bar{t})}{\Delta x}}. \quad (50)$$

(2) The above system consists of a tridiagonal system of linear algebraic equations. The solution x is the heat conductivity. A FORTRAN subroutine based on the Thomas algorithm [19] can be found in the popular heat conduction text by Ozisik [20] for solving a tridiagonal system of equations.

4. ERROR ANALYSIS OF THE NUMERICAL PROCEDURE

Understanding and controlling the numerical error is essential for a successful solution of the finite difference equation. In this section, a detailed error analysis will be presented from which we can conclude that the numerical procedure is at least second-order accurate.

By applying Taylor's expansion to equation (12), we have the following:

$$k_{j-1}^i \cdot (T_{j-1}^i - T_{j+1}^i) + 4k_j^i \cdot (T_{j+1}^i - 2T_j^i + T_{j-1}^i) + k_{j+1}^i \cdot (T_{j+1}^i - T_{j-1}^i) - 4\Delta x^2 \cdot \left(\frac{T_j^{i+1} - T_j^i}{\Delta t} - g_j^i \right) = O(\Delta x^3 + \Delta t \cdot \Delta x^2). \quad (51)$$

If we let $\Delta t = \Delta x^2$, then the local truncation error (LTE) within the interior points due to the finite differencing is $O(\Delta x^3)$.

Next, we analyze the LTE at the left boundary, R_1. By applying Taylor's expansion to equation (16), one may obtain the following equation:

$$k_0^i \cdot (T_2^i - 3T_1^i + 2T_0^i) + k_1^i \cdot (T_1^i - T_0^i) - \Delta x^2 \cdot \left(\frac{T_0^{i+1} - T_0^i}{\Delta t} - g_0^i \right) = O(\Delta x^2 + \Delta t \cdot \Delta x^2). \quad (52)$$

If we let $\Delta t = \Delta x^2$, the LTE at the R_1 boundary surface due to finite differencing is $O(\Delta x^2)$. A similar analysis can be performed to show that the local truncation error at the R_2 boundary surface is $O(\Delta x^2)$.

At this point, we have shown that the error of the discretization of the governing equation over the domain $R \bigcup R_1 \bigcup R_2$ is $O(\Delta x^2)$. Therefore, the system of linear equations (18) is different from the original equation by an error $O(\Delta x^2)$, i.e.

$$\mathbf{Ax} = \mathbf{b} + O(\Delta x^2). \quad (53)$$

By solving the system of linear equations (18), the solution $\mathbf{x}$ has an error $O(\Delta x^2)$. This shows the numerical procedure is second-order, which implies that the rate of convergence of the proposed procedure is second-order.

5. ILLUSTRATIVE NUMERICAL EXAMPLES

The finite difference procedure described above for the inverse determination of the thermal conductivity in a one-dimensional heat conduction domain was programmed in FORTRAN 77. The procedure is based on the computational algorithm given by equations (19)–(50) for the formulation. The computations were performed using a SUN Sparc10 Workstation. The numerical results were obtained in double-precision arithmetic.

To show the applicability of the proposed procedure, five distinct test cases were solved. Test cases include constant, spatially dependent, or temperature dependent quantities that are reconstructed from discrete temperature data. The heat conduction problems investigated are tabulated in Table 1.

The exact temperature and thermal conductivity used in the test case are pre-selected profiles that satisfy the governing heat conduction equation and the boundary conditions, as well as the initial condition. The simulated temperature data are generated from these pre-selected temperature profiles. The numerical procedure is tested by computing the thermal conductivity from these pre-selected temperatures and its accuracy is assessed by comparing the calculated results with the pre-selected thermal conductivity profiles.

As discussed above, the calculations are performed assuming that the temperature data are available at discrete grid points within the entire domain. The first three examples assume only the temperature data is available, hence the necessary condition $\partial T/\partial x = 0$ needs to be satisfied. Test cases 4 and 5 assume the heat fluxes are known at the left and right boundaries, respectively, in additional to the temperature data. The condition of $\partial T/\partial x = 0$ is not a requirement for these two cases.

Example 1—constant thermal conductivity

Consider a slab, $0 \leqslant x \leqslant 1$, with an initial temperature distribution which varies with the distance. For time $t > 0$, the boundaries at $x = 0$ and $x = 1$ are kept at zero temperature. To determine the thermal

Table 1. Heat conduction problems investigated

Example	Boundary conditions, $t > 0$: $x = 0$	Boundary conditions, $t > 0$: $x = 1$	Initial condition $T\|_{t=0}$	Heat generation $g(x, t)$	Temperature profile $T(x, t)$	Conductivity $k(x, t)$
1	$T = 0$	$T = 0$	$\sin(\pi x)$	0	$e^{-2\pi^2 t}\sin(\pi x)$	2
2	$T = 0.36te^{-t}$	$T = 0.16te^{-t}$	0	$(x-0.6)^2(1-t)e^{-t} - \{2+[0.5-4(x-0.3)(x-0.6)]e^{-4(x-0.3)^2}\}te^{-t}$	$(x-0.6)^2te^{-t}$	$1+0.25e^{-4(x-0.3)^2}$
3	$\frac{\partial T}{\partial x} = \pi e^{-\pi^2 t}\sin 0.8\pi$	$\frac{\partial T}{\partial x}+T = [\cos 0.2\pi - \pi\sin 0.2\pi]e^{-\pi^2 t}$	$\cos\pi(x-0.8)$	$-\pi^2 e^{-\pi^2 t}\cos\pi(x-0.8) + \frac{\pi^2 e^{-\pi^2 t}\cos\pi(x-0.8)}{1-e^{-\pi^2 t}\cos\pi(x-0.8)} - \left\{\frac{\pi e^{-\pi^2 t}\sin\pi(x-0.8)}{1-e^{-\pi^2 t}\cos\pi(x-0.8)}\right\}^2$	$e^{-\pi^2 t}\cos\pi(x-0.8)$	$\frac{1}{1-T(x,t)}$
4	$T = 9e^{-t}$ $q = -54e^{-t}$	$T = 4e^{-t}$	$(x-3)^2$	$-7(x-3)^2e^{-t}$	$(x-3)^2e^{-t}$	$(x-3)^2$
5	$T = 0$	$T = e^{-t}\sin(1)$ $q = 0.5[1+e^{-t}\sin(1)]e^{-t}\cos(1)$	$\sin(\pi x)$	$-0.5e^{-t}\sin(x) - 0.5e^{-2t}[\cos^2(x) - \sin^2(x)]$	$e^{-t}\sin(x)$	$0.5(1+T)$

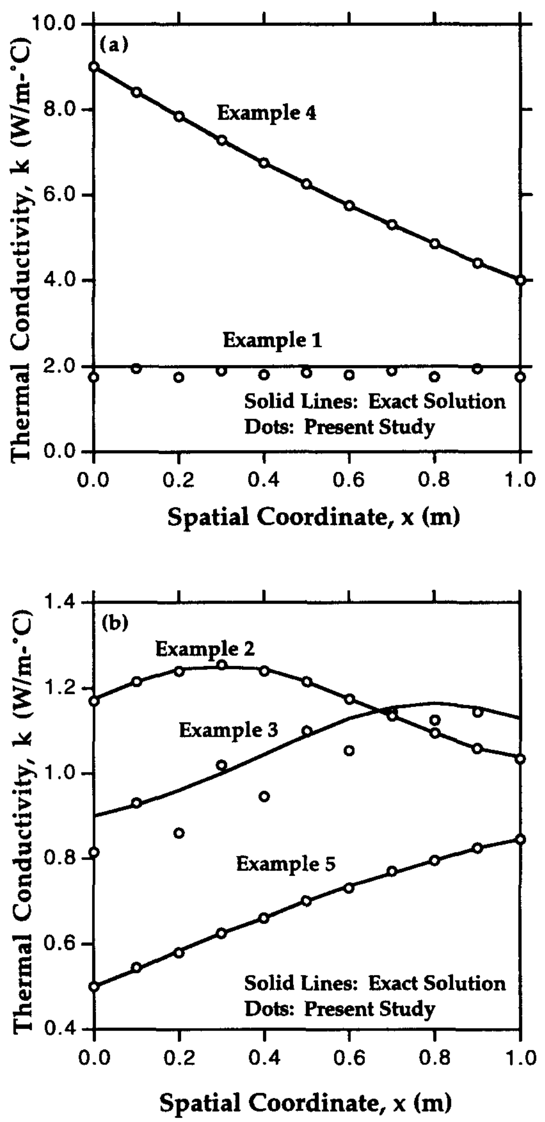

Fig. 2. Comparison of thermal conductivity with $\Delta x = 0.1$ and $t = 0.2$ for (a) examples 1 and 4; (b) examples 2, 3 and 5.

conductivity, the region $0 \leqslant x \leqslant 1$ is divided into $N = 10$, 20, and 40 intervals in the calculations which correspond to a mesh size of $\Delta x = 0.100$, 0.050 and 0.025, respectively. By comparing these predictions to the exact solutions for the conductivity, $k(x, t) = 2$, the maximum error corresponding to these runs are 0.2575, 0.0685 and 0.0174. These results clearly demonstrate that the numerical error decreased with the mesh size. Note that the maximum error also indicates that the convergent rate is $(\Delta x)^2$ and the numerical procedure is second-order accurate. The results of the thermal conductivities for mesh size $\Delta x = 0.1$ and $t = 0.2$ are presented in Fig. 2(a). The results from the present study are in very good agreement with the exact solution.

Example 2—spatially-dependent thermal conductivity

A plane wall, $0 \leqslant x \leqslant 1$, is initially at zero temperature. For time $t > 0$, heat is generated in the solid at a variable rate of $g(x, t)$, the boundaries at $x = 0$ and $x = 1$ are subjected to time-varied temperatures. To determine the thermal conductivity, the spatial coordinate $0 \leqslant x \leqslant 1$ is divided into $N = 10$, 20 and 40 intervals that correspond to $\Delta x = 0.100$, 0.050 and 0.025, respectively. The maximum errors for these corresponding cases are 0.0073, 0.0018 and 0.0004. Note that the accuracy of the prediction increases with decreasing grid size, as expected. The calculated thermal conductivity is compared with the exact function and the results are plotted in Fig. 2(b) for mesh size $\Delta x = 0.1$ and $t = 0.2$. Clearly, these numerical results are in excellent agreement. After examining the maximum errors for various mesh sizes, it is clear that the convergent rate of the proposed numerical procedure is second-order.

Example 3—temperature-dependent thermal conductivity

A slab, $0 \leqslant x \leqslant 1$, is initially maintained at a temperature which varies with distance. For time $t > 0$, heat is generated in the solid at a variable rate of $g(x, t)$. The derivative of temperature is prescribed at the boundary $x = 0$, while the boundary $x = 1$ dissipates heat by convection into an environment of zero temperature. Both conditions vary with time along the surfaces. Again, the spatial coordinate $0 \leqslant x \leqslant 1$ is divided into $N = 10$, 20 and 40 intervals in the calculations with a spacing of $\Delta x = 0.100$, 0.050 and 0.025. The corresponding maximum errors for these three runs were 0.1031, 0.0276 and 0.0070. The results are shown in Fig. 2(b) for $\Delta x = 0.1$ and $t = 0.2$. The inverse solutions for thermal conductivities at various times are shown in Fig. 3(a). The agreement between the calculated and exact values is very good.

Example 4—spatially-dependent thermal conductivity

A plane wall, $0 \leqslant x \leqslant 1$, is initially maintained at a prescribed temperature which varies with distance. For time $t > 0$, the boundaries at $x = 0$ and $x = 1$ are subjected to prescribed temperatures, which vary with time. In addition to the temperature measurements, the heat flux is also known at the left boundary, $x = 0$. Therefore, the condition $\partial T/\partial x = 0$ needs not be a requirement for the unique solution of the thermal conductivity. The calculated maximum errors on the estimated conductivity were found to be 0.0385, 0.0099 and 0.0025 that correspond to $\Delta x = 0.100$, 0.050 and 0.025, respectively. Again, as the mesh size decreases, the accuracy of the approximation increases. The calculated thermal conductivities from the present study are in good agreement with the analytical solutions and the comparison is shown in Fig. 2(a) for $\Delta x = 0.1$ and $t = 0.2$.

Example 5—temperature-dependent thermal conductivity

Consider a slab, $0 \leqslant x \leqslant 1$, that is initially at a prescribed temperature distribution and which varies

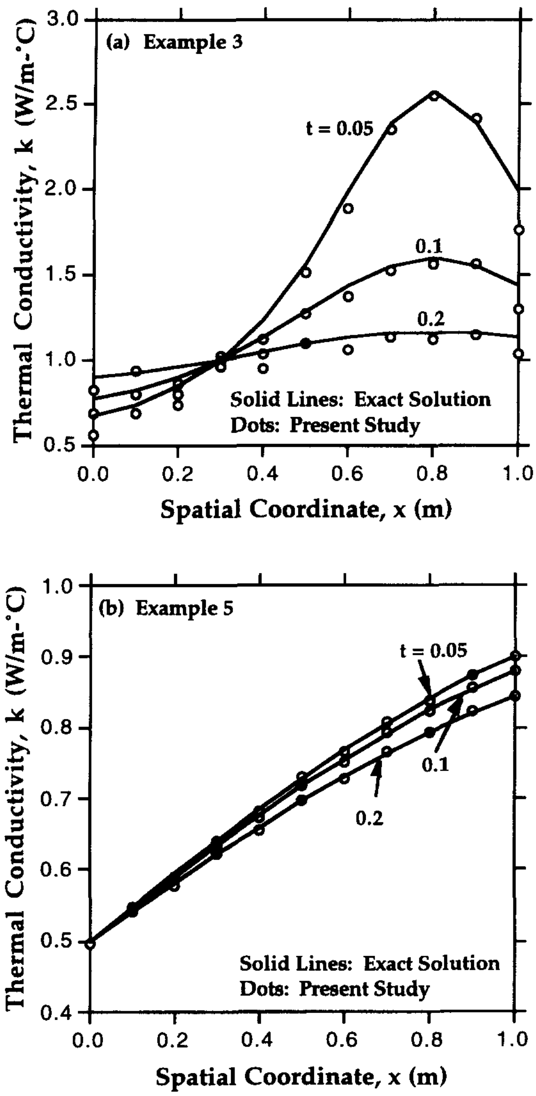

Fig. 3. Comparison of thermal conductivity at various time intervals for (a) example 3; (b) example 5.

with distance. For time $t > 0$, the boundary at $x = 0$ is subsequently kept at zero temperature, while the boundary surface at $x = 1$ is subjected to a prescribed temperature, which varies with time. In addition to the temperature measurements, the heat flux is known at the right boundary, $x = 1$. Therefore, the condition $\partial T/\partial x = 0$ needs not be a requirement for the unique solution of the thermal conductivity. The results are shown in Fig. 2(b) for $t = 0.2$ and $\Delta x = 0.1$. Figure 3(b) presents a comparison of thermal conductivities from the present study with the analytical solutions for various time intervals. Again the comparison clearly demonstrates the accuracy of the present method.

6. SENSITIVITY ANALYSIS

The simulated temperature data used in the inverse analysis of thermal conductivity were obtained from pre-selected temperature profiles. The error associated with the measurement of temperature would decrease the accuracy of the calculation of conductivity. Therefore, in an effort to quantify a realistic error in conductivity, the exact temperature inputs (T_{exact}) used here will be modified by adding random errors to simulate experimental measurements (T_{exp}) [21]

$$T_{exp} = T_{exact} + \varepsilon\sigma, \tag{54}$$

where σ is the standard deviation of the measurement error which is assumed to be the same for all measurements. For normally distributed errors with zero mean and a 99% confidence, the value of ε lies in the range

$$-2.576 < \varepsilon < 2.576. \tag{55}$$

The product of $\varepsilon\sigma$ represents the temperature measurement errors. In order to test the influence of the experimental errors on the inverse analysis, the

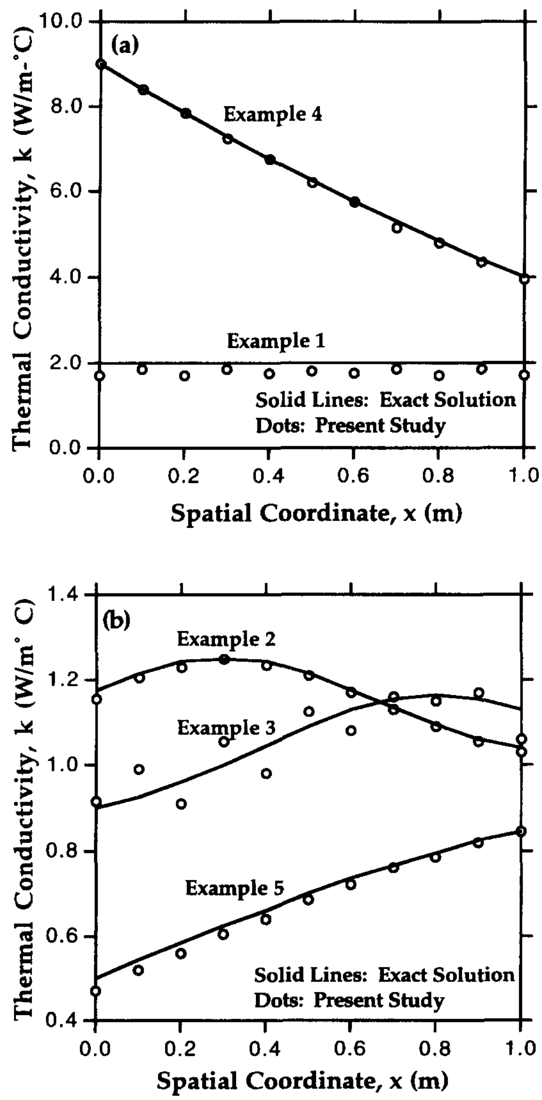

Fig. 4. Effect of temperature measurement error on the thermal conductivity with $\Delta x = 0.1$ and $t = 0.2$ for (a) examples 1 and 4; (b) examples 2, 3 and 5.

test cases described previously will be repeated by incorporating the measurement errors in the simulated temperature measurements. Under the most strict conditions, the simulated temperature data (T_{exp}) are generated by using the exact temperature profiles (Table 1) and equation (54) with $\sigma = 0.01\ (T_{exact})_{max}$ as well as $\varepsilon = 2.576$ or -2.576 in the inverse analysis.

The effects of temperature measurement error on the inverse analysis are shown in Fig 4(a, b). Clearly, the estimated thermal conductivities are in good agreement with the exact profiles.

7. CONCLUSIONS

A second-order finite difference procedure has been successfully developed for the inverse determination of the thermal conductivity of a one-dimensional medium. The unknown thermal conductivity is reconstructed using available temperature data or a combination of temperatures and surface heat flux. The estimated thermal conductivities were verified by comparing the results with the analytical solutions. The close agreement between the current results and the exact solutions confirms that the proposed finite difference scheme is an accurate technique for the inverse determination of thermal conductivities. The algorithm is straightforward and easy to implement and requires relatively little computer time for the computations.

A special feature of the approach is that no prior information is required about the functional form of the unknown conductivity. The algorithm is useful and attractive for heat transfer inverse analysis due to its simplicity, stability and high speed. The technique is applicable to linear and nonlinear spatially, as well as temperature-dependent thermal conductivities. Although the present algorithm is developed for the inverse analysis of one-dimensional heat transfer, it can be extended to solve two-dimensional geometries.

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